BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 13

SAM SCHIAVONE

CONTENTS

I. REVIEW

Last time we:

(1) Took a whirlwind tour of hyperbolic geometry in $\mathfrak{H}$ (and $\mathfrak{D}$).
   (a) The group of orientation-preserving isometries of $\mathfrak{H}$ is ________.
   (b) The geodesics of $\mathfrak{H}$ are ________________ that ___________ orthogonally.
(2) Defined Fuchsian groups.
   (a) A Fuchsian group is a _________ subgroup of $\mathrm{PSL}_2(\mathbb{R})$.
   (b) Fuchsian groups act ________________________ on $\mathfrak{H}$.
(3) Defined the notion of a fundamental domain for a Fuchsian group, and looked at some pictures of fundamental domains.

Definition 1. A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, i.e., a subgroup such that the subgroup topology is the discrete topology.

Proposition 2. *A subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian iff it acts properly discontinuously on $\mathfrak{H}$.*

Definition 3. Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$, and let D be a simply connected closed subset of $\mathfrak{H}$ whose boundary ∂D consists of a finite union of differentiable paths. D is a fundamental domain for Γ if $\{\gamma(D) : \gamma \in \Gamma\}$ tessellates $\mathfrak{H}$, i.e,

(1) $\bigcup_{\gamma\in\Gamma} \gamma(D) = \mathfrak{H}$; and
(2) for every $\gamma \in \Gamma \setminus \{1\}$, the intersection $D \cap \gamma(D)$ is contained in the boundary of D.

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Remark 4. In other words, any two translates of D don't intersect, except possibly on their boundaries. More formally,

$$D^\circ \cap (\gamma D)^\circ = \varnothing$$

for all $1 \neq \gamma \in \Gamma$, where D° denotes the interior of D.

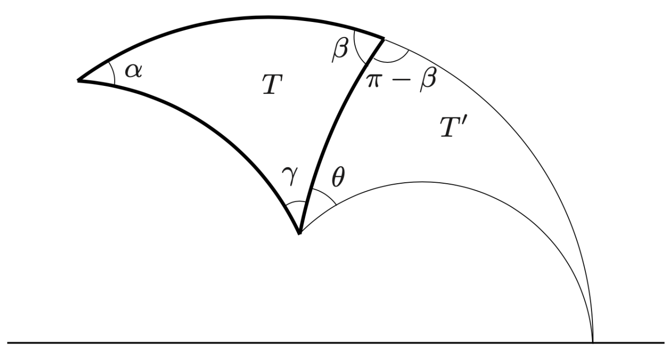
II. Fuchsian triangle groups

II.1. **Hyperbolic triangles and areas.** A hyperbolic triangle in $\mathfrak{H}$ is a topological triangle whose edges are hyperbolic geodesic segments. We allow the possibility of triangles with edges of infinite length, in which case at least one of the vertices lies in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

The characteristic property of hyperbolic spaces is the fact that the sum of the angles of a hyperbolic triangle is less than π.

Proposition 5. *If T is a hyperbolic triangle with angles α, β, γ, then the hyperbolic area of T is $a(T) = \pi - \alpha - \beta - \gamma$.*

Proof. We begin by making some reductions. First we show that it suffices to prove the result for triangles with at least one 0 angle. Assume we have proved the proposition for this case. If T is a triangle with angles α, β, γ, we can construct another triangle T' as below such that both T' and $T \cup T'$ are triangles with one 0 angle.

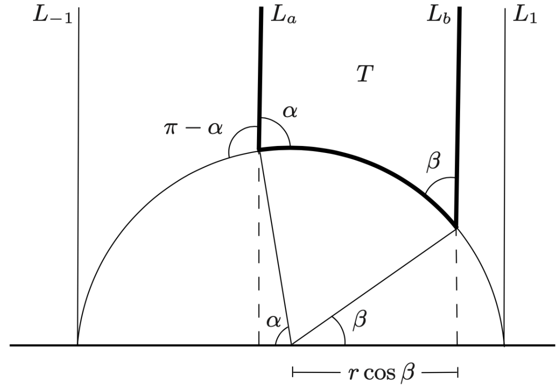


Then we know the result for the area of T' and $T \cup T'$, so

$$\begin{aligned} a(T) &= a(T \cup T') - a(T') = (\pi - \alpha - (\gamma + \theta) - 0) - (\pi - \theta - (\pi - \beta) - 0) \\ &= \pi - \alpha - \beta - \gamma, \end{aligned}$$

as desired.

Thus it remains to prove the result when one of the angles is 0. By applying an isometry of $\mathfrak{H}$, we can assume that one of the vertices is at ∞, and the two edges that intersect with angle 0 are vertical lines intersecting at ∞. Moreover, we can assume that the other edge is a segment of a circle centered at 0 of radius r.



One can calculate the area of this triangle directly by computing an integral, which I'll ask you to do for homework. □

II.2. **Reflections and rotations.** There are hyperbolic analogues of reflections and rotations. Lines are geodesics in Euclidean space, and just as we can define a reflection across a line, we can define a hyperbolic reflection across a geodesic.

Given a geodesic L in $\mathfrak{H}$, the reflection R_L over L is the unique nontrivial isometry fixing every point of L. For instance, the reflection across the imaginary axis L_0 is $R_0 : z \mapsto -\bar{z}$, and all other reflections are conjugate to this one. That is, given an arbitrary geodesic L in $\mathfrak{H}$, then $R_L = M \circ R_0 \circ M^{-1}$ where $M \in \mathrm{PSL}_2(\mathbb{R})$ is an isometry such that $M(L_0) = L$. Note that reflections are anticonformal—they preserve angles, but reverse orientation. Thus a reflection R is not holomorphic, but rather is antiholomorphic. In other words, R can be written in the form

$$R(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = -1$.

Just as for Euclidean space, the composition of two reflections is a rotation, and more specifically, if R_1 and R_2 are reflections fixing geoedesics L_1 and L_2, then $R_2 \circ R_1$ is a rotation about the point of intersection of L_1 and L_2, and if the angle between L_1 and L_2 is θ, then $R_2 \circ R_1$ is a rotation by 2θ.

For instance, the imaginary axis L_1 and the unit circle L_2 intersect at a 90° angle at i. As mentioned above, then $R_1 : z \mapsto -\bar{z}$ and $R_2 : z \mapsto 1/\bar{z}$, so

$$R_2 \circ R_1 : z \overset{R_1}{\mapsto} -\bar{z} \overset{R_2}{\mapsto} \frac{1}{(\overline{-\bar{z}})} = -1/z\,.$$

The corresponding matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}.$$

Remark 6. Although this looks like a rotation by $\pi/2$, it's actually a rotation by π. Note that the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix}$$

acts as the identity, since it sends $z \mapsto \dfrac{-z}{-1} = z$.

II.3. **Triangle groups.** Let $a, b, c \in \mathbb{Z}_{\geq 2}$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Then there is a hyperbolic triangle T with angles $\pi/a, \pi/b, \pi/c$. Denote the corresponding vertices by z_a, z_b, and z_c. Let L_a, L_b, L_c be the edge opposite z_a, z_b, z_c, and let τ_a, τ_b, τ_c be the reflections over the geodesic L_a, L_b, L_c, respectively. Let

$$\delta_a = \tau_c \tau_b \qquad \delta_b = \tau_a \tau_c \qquad \delta_c = \tau_b \tau_a$$

which are counterclockwise rotations about z_a, z_b, z_c by $2\pi/a, 2\pi/b, 2\pi/c$.

By repeatedly applying the reflections τ_a, τ_b, τ_c, we obtain a tessellation of $\mathfrak{D}$ by T. [Show picture on p. 118 of GGD.]

Proposition 7. *The triangle T is a fundamental domain for the group $\langle \tau_a, \tau_b, \tau_c \rangle$ generated by the reflections.*

Reflections are anti-holomorphic, so it's often easier to work with rotations, which are holomorphic. Letting $T^- = \tau_c(T)$, then we can also tessellate $\mathfrak{D}$ by repeatedly applying the rotations $\delta_a, \delta_b, \delta_c$ to the quadrilateral, which we call a triangle-pair or tri-pair for short,

$$Q = T \cup T^- = T \cup \tau_c(T)$$

comprised of the union of T and T^-.

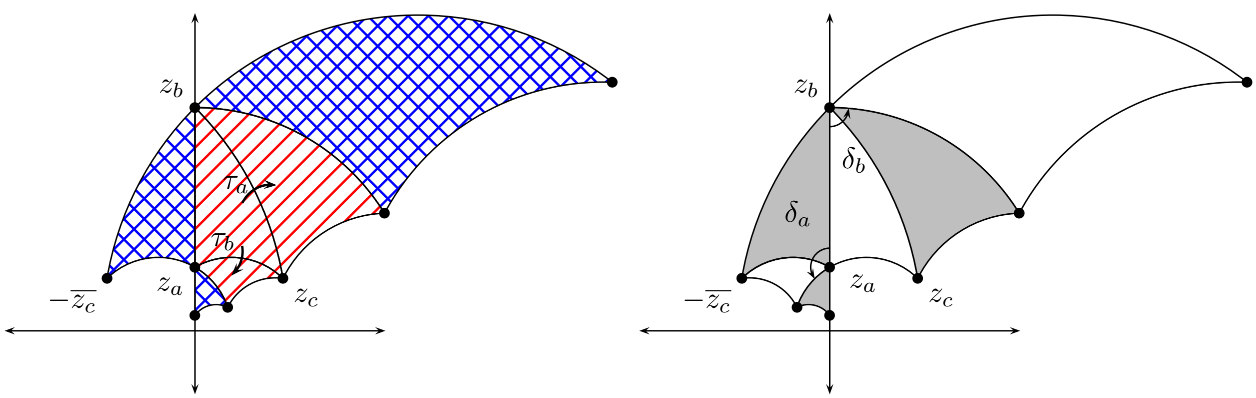


FIGURE 1. Hyperbolic reflections and rotations in $\mathfrak{H}$

Definition 8. Given $a, b, c \in \mathbb{Z}_{\geq 2}$ such that $1/a + 1/b + 1/c < 1$, the triangle group $\Delta(a, b, c)$ is the subgroup $\langle \delta_a, \delta_b, \delta_c \rangle$ of $\operatorname{Aut}(\mathfrak{D}) \cong \mathrm{PSL}_2(\mathbb{R})$.

Proposition 9. *The triangle-pair Q is a fundamental domain for $\Delta(a, b, c)$.*

Proposition 10. *The triangle group $\Delta(a, b, c)$ has presentation*

$$\langle \delta_a, \delta_b, \delta_c \mid {\delta_a}^a = {\delta_b}^b = {\delta_c}^c = \delta_a \delta_b \delta_c = 1 \rangle .$$

Remark 11. In the case where any of a, b, c are ∞, the "relation" $\delta_a^\infty = 1$ means no relation. For instance,

$$\Delta(\infty, \infty, \infty) = \langle \delta_a, \delta_b, \delta_c \mid \delta_a \delta_b \delta_c = 1 \rangle = \langle \delta_a, \delta_b \rangle$$

is the free group on two generators.

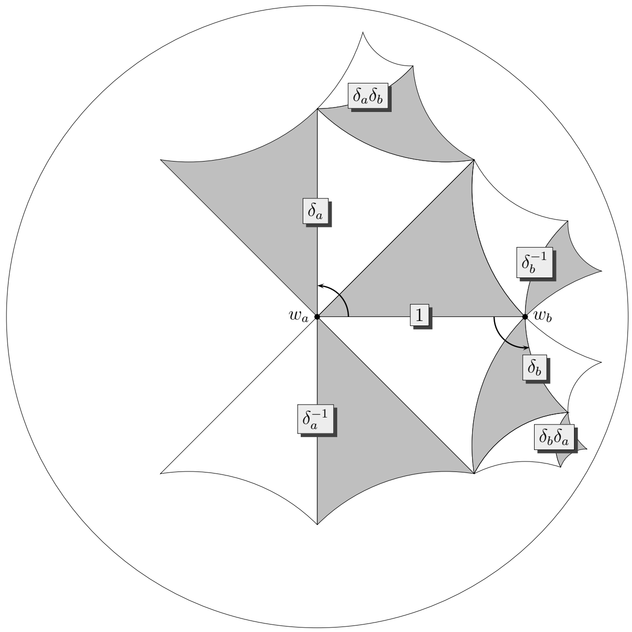


FIGURE 2. Hyperbolic rotations in $\mathfrak{D}$

One can show that the triangle group $\Delta = \Delta(a, b, c)$ acts properly discontinuously on $\mathfrak{D}$ for all choices of $a, b, c \in \mathbb{Z}_{\geq 2}$ such that $1/a + 1/b + 1/c < 1$. In other words, Δ is a Fuchsian group, so the quotient $\Delta\backslash\mathfrak{D}$ can be given the structure of a Riemann surface.

Proposition 12. $\Delta\backslash\mathfrak{D} \cong \mathbb{P}^1$, *as Riemann surfaces.*

Remark 13. In the case where some of a, b, c are ∞, then $\Delta\backslash\mathfrak{D}$ is isomorphic to $\mathbb{P}^1$ minus one, two, or three points.

Remark 14. When $1/a + 1/b + 1/c$ is > 1 or $= 1$, then instead the triangle T naturally lives on either the sphere or the Euclidean plane, rather than in hyperbolic space. One can similarly define a spherical or Euclidean triangle group in the same way. [Show picture on p. 120 of GGD.]

III. THE MODULAR GROUP AS A TRIANGLE GROUP

III.1. **The modular group.** Recall that a Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. One way to obtain such a group is to take a lattice L in $\mathbb{R}$, and then take $\mathrm{PSL}_2(L)$. In particular,

$$\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$$

is a Fuchsian group, called the modular group. We can show that this quite famous group (cf., modular forms) is actually a triangle group.

Let T be the hyperbolic triangle with vertices at $z_a = i$, $z_b = e^{2\pi i/6}$, and $z_c = \infty$ in $\mathfrak{H}$. The angles of T are $\pi/2$, $\pi/3$, and 0, respectively. As usual, let $Q = T \cup T^-$ be the union of T with its reflection across the imaginary axis. [Show picture on p. 121 of GGD.]

Two important elements of $\Gamma(1)$ are

$$T : z \mapsto z + 1 \qquad S : z \mapsto -1/z$$

which are represented by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 15. *Q is a fundamental domain for $\Gamma(1)$.*

Proof sketch. We'll just show the first property of being a fundamental domain, namely that every point $z \in \mathfrak{H}$ is $\Gamma(1)$-equivalent to some point in Q.

Given $z \in \mathfrak{H}$, by repeatedly applying the translation T or its inverse, we can move z into the vertical strip $-1/2 \leq \mathrm{Re}(z) \leq 1/2$. Replace z by the point with this property. If $|z| \geq 1$, then $z \in Q$, and we are done. Otherwise, $|z| < 1$ and applying S, we have

$$\mathrm{Im}(S(z)) = \mathrm{Im}(-1/z) = \mathrm{Im}(-\bar{z}/|z|^2) = \mathrm{Im}(z/|z|^2) > \mathrm{Im}(z).$$

Replace z by $-1/z$ and repeat the process. We claim that this algorithm terminates and produces a point inside Q that is $\Gamma(1)$-equivalent to our original z.

A straightforward computation shows that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2} \quad \text{for} \quad \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}).$$

Since there are only finitely many lattice points inside a disc, then there are only finitely many integer pairs $(c, d) \in \mathbb{Z}^2$ such that $|cz + d| < 1$. Thus there are only finitely many $\gamma \in \Gamma(1)$ such that γz has strictly larger imaginary part, which shows that the algorithm terminates. □

Corollary 16. $\Gamma(1) \cong \Delta(2, 3, \infty)$.

III.2. **Subgroups and congruence subgroups.**

Lemma 17. *Let Γ and Γ' be Fuchsian groups. Suppose that $\Gamma' \leq \Gamma$ and $[\Gamma : \Gamma'] = n$. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be a set of right coset representatives of $\Gamma' \backslash \Gamma$. Let Q be a hyperbolic polygon that is a fundamental domain for Γ. Then*

$$D := \bigcup_{j=1}^{n} \gamma_j(Q)$$

is a fundamental domain for Γ'.

Remark 18. In other words, if we know a fundamental domain Q for a Fuchsian group Γ and $\Gamma' \leq \Gamma$, we can obtain a fundamental domain for Γ' by translating Q by a set of coset representatives.

An important class of subgroups of $\Gamma(1)$ are so-called principal congruence subgroups. For $N \in \mathbb{Z}_{\geq 1}$, the principal congruence subgroup $\Gamma(N)$ is the kernel of the reduction map

$$\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$$

that reduces the entries of a matrix modulo N. In other words, $\Gamma(N)$ fits into a short exact sequence

$$1 \to \Gamma(N) \to \Gamma(1) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1.$$

$\Gamma(2)$ turns out to be of particular interest. We will show that $\Gamma(2) \cong \Delta(\infty, \infty, \infty)$, the free group on two generators.